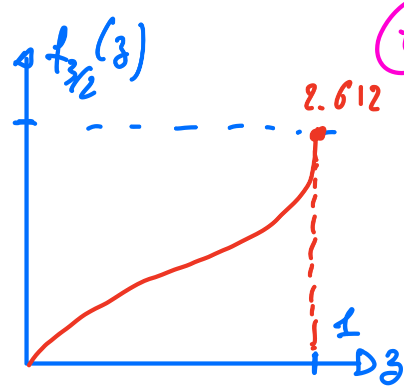
$$\rho_0 = \underbrace{\frac{g}{V}\,\frac{z}{1-z}}_{\rho_{GS}(z)} + \underbrace{g\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} f_{3/2}(z)}_{\rho_{ES}(z) < \rho_{ES}^{MAX} = \frac{g}{\lambda^3}\, 2.612}$$



Canonical perspective For large systems, the description of intensive quantities like ρ & μ is expected to be equivalent in all ensembles. We can thus think about fixing ρ_0 & solving $\rho_0 = \rho_{GS}(z) + \rho_{ES}(z)$ to get z, ρ_{GS} & ρ_{ES}.

If $\rho_0 < \rho_{ES}^{MAX}$, we can find $z < 1$ such that

$$\rho_0 = \underbrace{\rho_{GS}}_{\propto \frac{1}{V} \xrightarrow[V\to\infty]{} 0} + \rho_{ES}(z) \Rightarrow \rho_0 \simeq \rho_{ES}(z) \Rightarrow \text{No BEC}$$

For $\rho_0 > \rho_{ES}^{MAX}$, this is impossible & we have condensation.

Comment: when there is condensation, we denote $\alpha = \frac{\rho_{GS}}{\rho_0}$ the fraction of particle in the ground state

* $\langle n_0 \rangle = \alpha V \rho_0 = \frac{g}{z^{-1}-1} \Rightarrow z^{-1} = 1 + \frac{g}{\alpha \rho_0 V} \Rightarrow$ recover $z-1 \propto \frac{1}{V}$ scaling
* What about $\langle n_1 \rangle = \frac{1}{z^{-1} e^{\beta \varepsilon_1} - 1}$? $\vec{k}_1 = \frac{2\pi}{L} \Rightarrow \beta \varepsilon_1 = \beta \frac{\hbar^2}{2m}\frac{4\pi^2}{L^2} \equiv \frac{K}{L^2}$

$$\langle n_1 \rangle \simeq \left[\left(1 + \frac{g}{\alpha \rho_0 V}\right)\left(1 + \frac{K}{L^2}\right) - 1\right]^{-1} \simeq \left[\frac{K}{L^2} + \frac{1}{\alpha \rho_0 L^3}\right]^{-1}$$

$$\boxed{\rho_1 = \frac{\langle n_1 \rangle}{V} \simeq \frac{L^2}{K L^3} \xrightarrow[L\to\infty]{} 0}$$

$\Rightarrow$ only the ground state has an extensive number of particles.

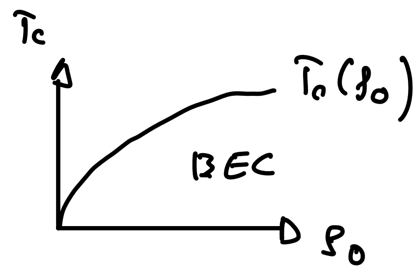
Transition temperature

μ, z or N are not the easiest parameter to control, but we can tune $\rho_{ES}^{MAX} = g\left(\frac{2\pi m k T}{h^2}\right)^{3/2} f^{+}_{3/2}(1)$ by changing T.

* $T > T_c$, $\rho_0 < \rho_{ES}^{MAX}(T)$ so that $\rho_0 = \rho_{GS}(z) + \rho_{ES}(z)$ admits a solution with $z < 1 \Rightarrow \rho_{GS} \xrightarrow[V\to\infty]{} 0$

* At T_c, $\rho_0 = \rho_{ES}^{MAX}(T_c) \Rightarrow \boxed{k T_c = \frac{h^2}{2\pi m}\left(\frac{\rho_0}{g\, f_{3/2}(1)}\right)^{2/3}}$

$\rho_0 = \rho_{ES}^{MAX}(1) \Rightarrow z \to 1$ & $\mu = 0$

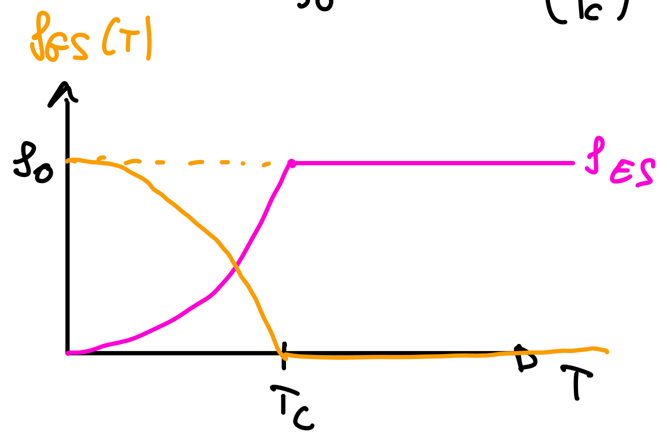


* $T < T_c$, z & μ stuck at $z = 1$ & $\mu = 0$

$$\rho_{GS} = \rho_0 - \underbrace{\rho_{ES}^{MAX}(T)}_{\propto T^{3/2}} \Rightarrow \frac{\rho_{GS}}{\rho_0} = 1 - \frac{\rho_{ES}^{MAX}(T)}{\rho_0} \Rightarrow BEC$$

Since T_c such that $\rho_0 = \rho_{ES}^{MAX}(T_c) \Rightarrow \frac{\rho_{GS}}{\rho_0} = 1 - \left(\frac{T}{T_c}\right)^{3/2}$

Phase diagram in the canonical ensemble



For more detail on canonical vs grand canonical: [arXiv: 2404.17300].

Thermodynamics

Grand potential

Again, we treat the GS separately

$$G = g k_B T \sum_{\vec{k}} \ln\left[1 - e^{\beta(\mu - \varepsilon(\vec{k}))}\right]$$

$$G = k_B T g \left[\ln(1-z) + \frac{V}{(2\pi)^3} \int d\vec{k} \ln\left(1 - z e^{-\beta \frac{\hbar^2 k^2}{2m}}\right) \right]$$

$$x = \frac{\hbar^2 k^2}{2 m k_B T} \Rightarrow k = \sqrt{x} \sqrt{\frac{8\pi^2 m k_B T}{h^2}}$$

$$G = k_B T g \ln(1-z) + \frac{g V k_B T}{4\pi^2} \left(\frac{8\pi^2 m k_B T}{h^2}\right)^{\frac{3}{2}} \underbrace{\int dx \; x^{1/2} \ln(1 - z e^{-x})}_{\text{IBPs}: -\frac{2}{3} \int dx \frac{x^{3/2} z e^{-x}}{1 - z e^{-x}}}$$

$$G = k_B T g \ln(1-z) - \frac{g V k_B T}{\Lambda^3} \underbrace{\frac{2}{\sqrt{\pi}} \cdot \frac{2}{3}}_{1/(3/2)!} \int dx \frac{x^{3/2}}{z^{-1} e^{x} - 1}$$

$$G = k_B T g \ln(1-z) - \frac{g V k_B T}{\Lambda^3} f^+_{5/2}(z)$$

Pressure $\quad P = -\frac{\partial G}{\partial V} = \frac{g k_B T}{\Lambda^3} f_{5/2}(z) \Rightarrow$ the ground state bosons do not contribute to the pressure

This makes sense: $\vec{k}_0 = \vec{0} \Rightarrow$ no momentum $\vec{p} = \hbar \vec{k} \Rightarrow$ no contribution to pressure.

$T < T_c$ $\quad z = 1;\ f_{5/2}(1) \simeq 1.31$

$$\Rightarrow P(T < T_c) \simeq 1.31 \frac{g k_B T}{\Lambda^3} \Rightarrow \text{independent of } N \,\&\, V!$$

$T > T_c$: $\quad \rho_0 = \frac{g}{\Lambda^3} f_{3/2}(z) \Rightarrow P = \rho_0 k T \frac{f^+_{5/2}(z)}{f^+_{3/2}(z)}$

$$T \gg T_c \Rightarrow z \ll 1;\ f^+_n(z) = \frac{1}{(n-1)!} \int_0^\infty dx \frac{x^{n-1}}{z^{-1} e^{x} - 1} \simeq \frac{z}{(n-1)!} \underbrace{\int_0^\infty dx\, x^{n-1} e^{-x}}_{(n-1)!}$$

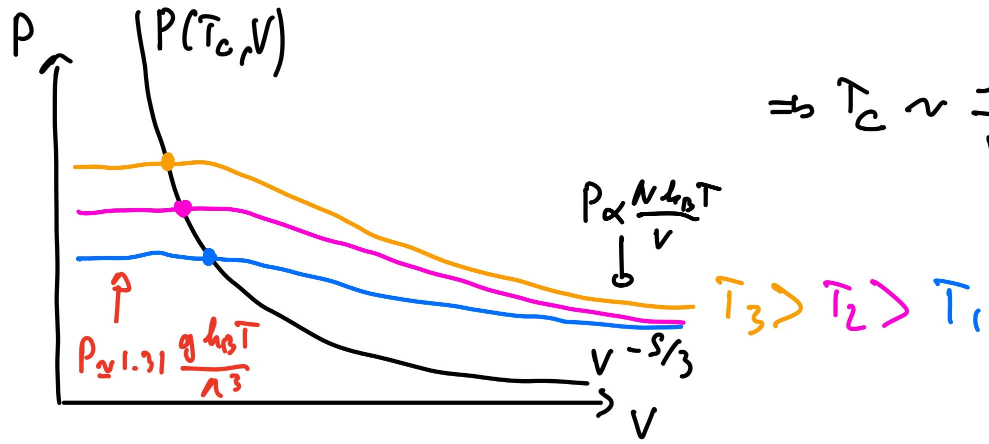
$$\simeq z$$

$$\Rightarrow P(T \gg T_c) = \rho_0 k T \text{ as expected}$$

Full isotherm: P(V) at different T

$$\rho_0 = \frac{N}{V} = \frac{g\, f^+_{3/2}(1)}{h^3}\left(2\pi m k_B T_c\right)^{3/2} \Rightarrow T_c = \frac{1}{2\pi m k_B}\left[\frac{N h^3}{V g\, f^+_{3/2}(1)}\right]^{2/3}$$

$$\Rightarrow T_c \sim \frac{1}{V^{2/3}} \Rightarrow P(T_c) \sim \rho_0 T_c \sim \frac{1}{V^{5/3}}$$



High temperature expansion:

$f^+_m(z) \simeq z$ is only the leading term in the small z / high T expansion $\Rightarrow$ go to higher order

$$f^+_m(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^m} \simeq z + \frac{z^2}{2^m} + \frac{z^3}{3^m}$$

From there: P as series in z; n as series in $z \Rightarrow z$ as series in n } P as series in n

Proof: $f_m(z) = \frac{1}{(m-1)!}\int_0^\infty dx\, x^{m-1} \frac{z e^{-x}}{1-z e^{-x}} = \frac{1}{(m-1)!}\int_0 dx\, x^{m-1} z e^{-x} \sum_{k=0}^{\infty} (z e^{-x})^k$

$$= \frac{1}{(m-1)!}\sum_{k=0}^{\infty} z^{k+1} \underbrace{\int_0^\infty dx\, x^{m-1} e^{-(k+1)x}}_{\frac{(m-1)!}{(k+1)^m}} \qquad u = (k+1)x$$

Energy & heat capacity:

$$\langle E\rangle = \partial_\beta(\beta G) = \frac{3gV}{\lambda^4} f_{5/2}(z) \frac{\partial\lambda}{\partial\beta} \quad ; \lambda = \sqrt{\frac{h^2\beta}{2\pi m}} \Rightarrow \partial_\beta \lambda = \frac{1}{2\beta}\lambda = \frac{kT}{2}\lambda$$

$$\boxed{\langle E\rangle = \frac{3}{2} kT \frac{gV}{\lambda^3} f_{5/2}(z) = \frac{3}{2} PV}$$

How does this compare with classical stat mech? Let's reparametrize

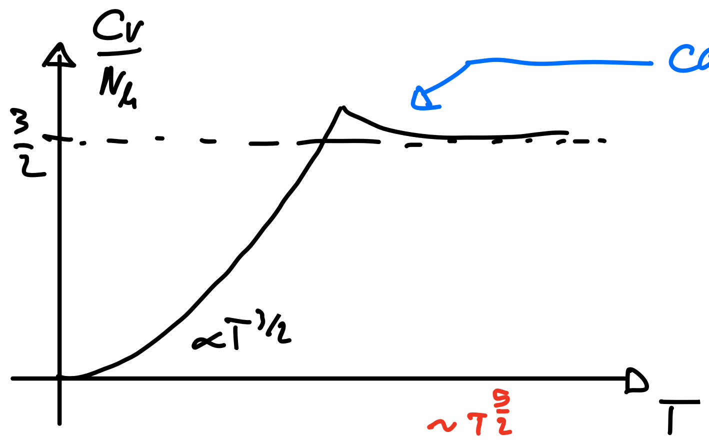
$T>T_c$, $N = \frac{gV}{\Lambda^3} f^+_{3/2}(z) \Rightarrow E = \frac{3}{2} NkT \frac{f_{5/2}(z)}{f_{3/2}(z)}$, $T \gg T_c$ $E = \frac{3}{2} NkT$

$C_V = \frac{3}{2} N$

At $T=T_c$, $\rho_0 = \frac{g}{\Lambda_c^3} f^+_{3/2}(1) \Rightarrow gV = \frac{N\Lambda_c^3}{f_{3/2}(1)}$

$T<T_c$, $\langle E\rangle = \frac{3}{2} kTN \left(\frac{\Lambda_c}{\Lambda}\right)^3 \frac{f^+_{5/2}(1)}{f^+_{3/2}(1)} = \frac{3}{2} kTN \left(\frac{T}{T_c}\right)^{3/2} \frac{f_{5/2}(1)}{f_{3/2}(1)}$

$\Rightarrow \langle E\rangle \propto T^{5/2} N$ & $C_V \propto N T^{3/2}$



can we compute this?

$T \geq T_c$; $\langle E\rangle = \frac{3}{2} k \overbrace{T \frac{gV}{\Lambda^3}}^{\sim T^{\frac{5}{2}}} f^+_{5/2}(z) \Rightarrow C_V = \frac{3}{2} kT \frac{gV}{\lambda^3}\left[\frac{5}{2T} f^+_{5/2}(z) + \underbrace{\partial_z f^+_{5/2}(z)}_{①} \cdot \underbrace{\frac{\partial z}{\partial T}}_{②}\right]$

① Direct algebra $\partial_z f^+_{5/2}(z) = \frac{1}{z} f^+_{3/2}(z)$

$\partial_x\left[-\frac{1}{z^{-1}e^x-1}\right]$

$$\partial_z \int_0^\infty dx \frac{x^{m-1}}{z^{-1}e^x-1} = -\int_0^\infty dx \frac{x^{m-1}}{(z^{-1}e^x-1)^2}\left(-\frac{1}{z^2}e^x\right) = \frac{1}{z}\int_0^\infty dx \frac{z^{-1}e^x}{(z^{-1}e^x-1)^2} x^{m-1}$$

$$= \frac{m-1}{z}\int_0^\infty dx \frac{x^{m-2}}{z^{-1}e^x-1}$$

$\times \frac{1}{m!} \Rightarrow \partial_z f^+_m = \frac{1}{z} f^+_{m-1}$

② $\rho_0 = \frac{g f_{3/2}(z)}{\Lambda^3}$. At fixed ρ_0, $\partial_T \ln \rho_0 = 0$

$$\partial_T \ln f_{3/2}(z) = 3\partial_T \ln\Lambda = -\frac{3}{2T} = \partial_z \ln f_{3/2}(z) \cdot \frac{\partial z}{\partial T} = \frac{\partial_z f_{3/2}(z)}{f_{3/2}(z)} \frac{\partial z}{\partial T} = \frac{1}{z}\frac{f^+_{1/2}(z)}{f^+_{3/2}(z)} \partial_T z$$

$$\Rightarrow \partial_T z = -\frac{3z}{2T}\,\frac{f^+_{3/2}}{f^+_{1/2}}$$

$$\Rightarrow \boxed{C_V = \frac{15}{4} k_B \frac{gV}{\Lambda^3} f^+_{5/2}(z) - \frac{9}{4} k_B \frac{gV}{\Lambda^3} \frac{f^2_{3/2}}{f_{1/2}}}$$

When $T \to T_c$ & $z \to 1$, $\left.\begin{array}{l} f^+_{5/2} \to 1.34 \\ f^+_{3/2} \to 2.61 \\ f^+_{1/2} \to +\infty \end{array}\right\}$ $C_V \simeq \frac{15}{4} k_B \overbrace{\frac{gV}{\Lambda^3}}^{\frac{N}{f_{3/2}(1)}} \Rightarrow \frac{C_V}{N k_B} \simeq 1.92 > \frac{3}{2}$

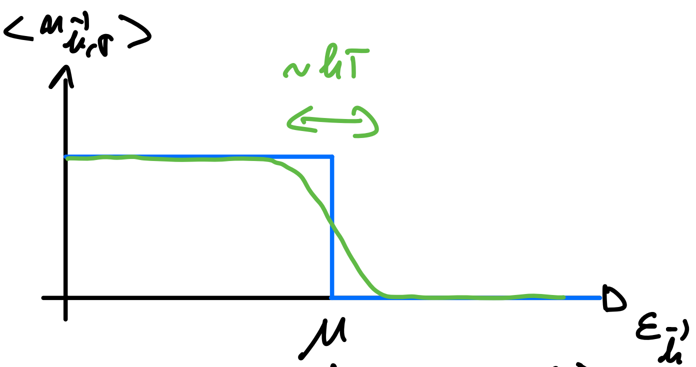
High temperature expansion: $\frac{C_V}{N k_B} \simeq \frac{3}{2}\left(1 + \rho_0 \frac{\Lambda^3}{2^{7/2}} + \dots\right)$

7. Fermi-Dirac Statistics

$$\langle n_{\vec{k},\sigma} \rangle = \frac{1}{e^{\beta[\varepsilon_{\vec{k}} - \mu]} + 1}$$

$$\frac{1}{1+e^x} \begin{cases} \to 0 & \text{as } x \to +\infty \\ \to 1 & \text{as } x \to -\infty \end{cases}$$

Statistics at low temperature



At $t=0$, all levels are full up to $\varepsilon_{\vec{k}} = \mu$ and empty above μ. $\varepsilon_F = \mu$ is then called the Fermi energy.

The occupied levels are called the Fermi sea. They satisfy

$$\varepsilon_k < \varepsilon_F \Leftrightarrow \frac{\hbar^2 k^2}{2m} < \varepsilon_F$$

$\Rightarrow k_F = \sqrt{\frac{2m\mu}{\hbar^2}}$ is the Fermi wavenumber